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Research Article

On the Nonlinear Perturbation $K(n, m)$ Rosenau-Hyman Equation: A Model of Nonlinear Scattering Wave

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We investigate a nonlinear wave phenomenon described by the perturbation $K(n, m)$ Rosenau-Hyman equation within the concept of derivative with fractional order. We used the Caputo fractional derivative and we proposed an iteration method in order to find a particular solution of the extended perturbation equation. We proved the stability and the convergence of the suggested method for solving the extended equation without any restriction on (m, n) and also on the perturbations terms. Using the inner product we proved the uniqueness of the special solution. By choosing randomly the fractional orders and m, we presented the numerical solutions.

1. Introduction

Fractional calculus is referring to calculus of integrals and derivatives with any real or complex order and has increased significantly attractiveness throughout the past three decades, owing for the most part to its established relevance in many fields of science and engineering. It does without a doubt make quite a lot of potentially advantageously helpful equipment available for finding the solutions of differential and integral equations and further problems connecting exceptional functions of mathematics physics in addition to their additional rooms and generalizations in one and more variables. No wonder researchers nowadays are interested in modelling several physical problems within the scope of the fractional calculus. This leads us to have a look at the nonlinear scattering waves.

Generalized Korteweg-de Vries equations with nonlinear dispersion can propagate compactly supported solitary waves, referred to as compactons [1–7]. Numerical simulations give you an idea about an original pulse wider than a compacton with small amount of radiation; in addition compactons collide elastically suffering only a phase shift after the collision and generating a small amplitude, zero-mass, compact (see [8–13]). Primarily discovered in the $K(n, m)$ Rosenau-Hyman equation for the modeling of pattern developments in liquid drops, compactons have more than a few functions in physics and science [1], for instance, pattern configuration on liquid plane [14]. The $K(n, m)$ equation is in addition the permanent boundary of the disconnected equations of a nonlinear lattice [1–15] and has been generalized to advanced dimensions [16]. Let us also note that the $K(n, m)$ equation has other kinds of solutions, for instance, the elliptic compactons [1, 17, 18]. At long last, several generalizations of the $K(n, m)$ equations have also been well thought-out in the literature, such as the insertion of time-dependent damping and dispersion [19] or the addition of fifth-order dispersion [20]. In this work we are very much interested in investigating the special solution of the perturbation $K(n,m)$ Rosenau-Hyman equation, which we will generalize as follows:

$$\begin{aligned}\partial_t^{\alpha} u + \partial_x u^m + \partial_{xxx}^3 u^n = &-\mu_0 \partial_{xxxxxx}^6 u - \mu_1 \partial_{xxxxxt}^6 u \\ &- \mu_2 \partial_{xxxxtt}^6 u - \mu_3 \partial_{xxxttt}^6 u \\ &- \mu_4 \partial_{xxtttt}^6 u - \mu_5 \partial_{xttttt}^6 u \\ &- \mu_6 \partial_{tttttt}^6 u,\end{aligned} \quad (1)$$

where $|\mu_j| \ll 1$ are small parameters. The above equation will be called the perturbation fractional $K(n,m)$ Rosenau-Hyman equation. Because of the plentiful profits offered by these fractional derivatives, many researchers have been stressing on proposing new definitions of fractional derivatives. We shall present the brief summary of these derivatives in the following section.

2. Some Fractional Calculus Definitions

It is perhaps important to mention that the concept of noninteger order derivative is an older concept, since it is considered to have stanched after an interrogation rose in the year 1695 by Marquis de L'Hopital. We shall present some of these definitions here.

Definition 1 (see [21–26]). The Riemann-Liouville fractional derivative is as follows: according to Riemann-Liouville the fractional derivative of a function says f is given as

$$D_t^{\alpha}(f(t)) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-x)^{n-\alpha-1} f(x)\,dx, \quad n-1 < \alpha \le n. \quad (2)$$

Definition 2. The Riemann-Liouville fractional integral is as follows: according to Riemann-Liouville, the fractional integral that is considered as antifractional derivative of a function f is given as

$$I_t^{\alpha}(f(t)) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x)\,dx, \quad x > a. \quad (3)$$

Definition 3. Caputo fractional derivative is as follows: according to Caputo, the fractional derivative of a continuous and n-time differentiable function f is given as

$$D_t^{\alpha}(f(t)) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{n-\alpha-1} \left(\frac{d}{dx}\right)^n f(x)\,dx, \quad n-1 < \alpha \le n. \quad (4)$$

Definition 4. The modified Riemann-Liouville fractional derivative of a function f is given as

$$D_t^{\alpha}(f(t)) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-x)^{n-\alpha-1} [f(x) - f(a)]\,dx, \quad n-1 < \alpha \le n. \quad (5)$$

There are other definitions that are not mentioned here. We shall present the derivation of the special solution in the next section. However, in this paper we shall use the Caputo type.

3. Derivation of the Special Solution

One of the great challenges in the field of partial of differential equation is to derive the solution of especially nonlinear equations, not to mention nonlinear equations described with the fractional order derivative, for example, the perturbation fractional $K(n,m)$ Rosenau-Hyman equation that has stronger nonlinearity. No wonder scholars of this area are always trying to find suitable and easier methods to derive at least approximate solution of these difficult equations. In this paper we will make use of a simple iteration method to propose a special solution to the perturbation fractional $K(n,m)$ Rosenau-Hyman equation; the methods is called the new variational iteration method (NVIM) [27], which uses the idea of Lagrange multiplier. We shall first show the methodology to accommodate readers that are not used to this method. Now consider a general partial differential equation with high order (m) with respect to time; then, in this new VIM, the first step is to apply the Laplace transform on both sides of equation to obtain

$$\begin{aligned}&s^m w(x,s) - s^{m-1} w(x,0) - \cdots w^{m-1}(x,0) \\ &\quad = \mathscr{L}[L(w(x,t)) + N(w(x,t)) + k(x,t)].\end{aligned} \quad (6)$$

The recursive formula of (6) can now be used to put forward the main recursive method connecting the Lagrange multiplier as

$$\begin{aligned}w_{n+1}(x,s) = &w_n(x,s) + \lambda(s)\left[s^m w_n(x,s)\right. \\ &- s^{m-1} w(x,0) - \cdots w^{m-1}(x,0) \\ &\left.- \mathscr{L}\left[L(w_n(x,t)) + N(w_n(x,t)) + k(x,t)\right]\right].\end{aligned} \quad (7)$$

Now considering $\mathscr{L}[L(w_n(x,t)) + N(w_n(x,t)) + k(x,t)]$ as the restricted term, the Lagrange multiplier can be obtained as [27]

$$\lambda(s) = -\frac{1}{s^m}. \quad (8)$$

Now applying the inverse Laplace transform on both sides of (6), we obtain the following iteration:

$$\begin{aligned}w_{n+1}(x,t) = &w_n(x,t) - \mathscr{L}^{-1}\left[\frac{1}{s^m}\left[-\mathscr{L}\left[L(w_n(x,t))\right.\right.\right. \\ &\left.\left.\left.+ N(w_n(x,t)) + k(x,t)\right]\right]\right]\end{aligned} \quad (9)$$

with the first term to be

$$\begin{aligned}w_0(x,t) = &-\mathscr{L}^{-1}\left[\frac{1}{s^m}\left(s^m w_n(x,s) - s^{m-1} w(x,0)\right.\right. \\ &\left.\left.- \cdots w^{m-1}(x,0)\right)\right].\end{aligned} \quad (10)$$

Now following the above methodology, we can obtain the Lagrange multiplier of (1) to be

$$\lambda(s) = -\frac{1}{s^{\alpha}} \quad (11)$$

and the iteration formula associate is given by

$$u_{k+1} = \mathscr{L}^{-1}\left(-\frac{1}{s^{\alpha}}\mathscr{L}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-l)^{\alpha-1} \cdot \left(-\partial_x u^m{}_k - \partial^3_{xxx}u^n{}_k - \mu_0\partial^6_{xxxxxx}u_k - \mu_1\partial^6_{xxxxxl}u_k - \mu_2\partial^6_{xxxxll}u_k - \mu_3\partial^6_{xxxlll}u_k - \mu_4\partial^6_{xxllll}u_k - \mu_5\partial^6_{xlllll}u_k - \mu_6\partial^6_{llllll}u_k\right) dl\right)\right) \quad (12)$$

and then

$$u(x,t) = \lim_{k\to\infty} u_{k+1}. \quad (13)$$

Theorem 5 (convergence analysis). *Let us consider*

$$\begin{aligned} T(u) &= \partial^{\alpha}_t u \\ &= -\partial_x u^m - \partial^3_{xxx}u^n - \mu_0\partial^6_{xxxxxx}u - \mu_1\partial^6_{xxxxxt}u \\ &\quad - \mu_2\partial^6_{xxxxtt}u - \mu_3\partial^6_{xxxttt}u - \mu_4\partial^6_{xxtttt}u \\ &\quad - \mu_5\partial^6_{xttttt}u - \mu_6\partial^6_{tttttt}u \end{aligned} \quad (14)$$

and consider the initial and boundary condition for (1); then the new variation iteration method leads to a special solution of (1).

Proof. To achieve this we shall think about the following fractional sub-Hilbert space H_α of the Hilbert space $H = L^2((a,b)\times(0,T))$ [28] that can be defined as the set of those functions:

$$\begin{gathered} v\colon (a,b)\times[0,T] \longrightarrow \mathbb{R}, \\ \frac{1}{\Gamma(\alpha)}\iint (t-l)^{\alpha-1} v^m dl\, ds < \infty. \end{gathered} \quad (15)$$

We harmoniously assume that the differential operators are restricted under the L^2 norms. Using the definition of the operator, T, we have the following:

$$\begin{aligned} T(u) - T(v) = &-\partial_x(u^m - v^m) - \partial^3_{xxx}(u^n - v^n) \\ &- \mu_0\partial^6_{xxxxxx}(u-v) \\ &- \mu_1\partial^6_{xxxxxt}(u-v) \\ &- \mu_2\partial^6_{xxxxtt}(u-v) - \mu_3\partial^6_{xxxttt}(u) - v \\ &- \mu_4\partial^6_{xxtttt}(u-v) - \mu_5\partial^6_{xttttt}(u-v) \\ &- \mu_6\partial^6_{tttttt}(u-v). \end{aligned} \quad (16)$$

Our next task is to evaluate $(T(u)-T(v), u-v)$ with $(,)$ being the inner product. Then, we have the following:

$$\begin{aligned} (T(u)-T(v), u-v) = &-(\partial_x(u^m - v^m), u-v) \\ &-(\partial^3_{xxx}(u^n - v^n), u-v) \\ &-\mu_0(\partial^6_{xxxxxx}(u-v), u-v) \\ &-\mu_1(\partial^6_{xxxxxt}(u-v), u-v) \\ &-\mu_2(\partial^6_{xxxxtt}(u-v), u-v) \\ &-\mu_3(\partial^6_{xxxttt}(u-v), u-v) \\ &-\mu_4(\partial^6_{xxtttt}(u-v), u-v) \\ &-\mu_5(\partial^6_{xttttt}(u-v), u-v) \\ &-\mu_6(\partial^6_{tttttt}(u-v), u-v). \end{aligned} \quad (17)$$

To evaluate the above expression, we shall consider case by case: we shall start with the following:

$$(\partial_x(u^m - v^m), u-v). \quad (18)$$

With the benefit of the Cauchy-Schwartz inequality, we have the following relation

$$(\partial_x(u^m - v^m), u-v) \le \|\partial_x(u^m - v^m)\|\,\|u-v\|. \quad (19)$$

The most important part in the above inequality is $\|\partial_x(u^m - v^m)\|$; let us handle this part; first making use of the continuity properties of the derivative, it is possible for us to find a positive constant θ_1 such that

$$\begin{aligned} \|\partial_x(u^m - v^m)\| &\le \theta_1 \|u^m - v^m\|, \\ u^m - v^m &= (u-v)\left(\sum_{j=0}^{m-2} C^j_{m-1}u^j v^{m-j-2}\right). \end{aligned} \quad (20)$$

Therefore

$$\|\partial_x(u^m - v^m)\| \le \theta_1 \|u-v\| \left\|\sum_{j=0}^{m-2} C^j_{m-1}u^j v^{m-j-2}\right\|. \quad (21)$$

Now if one makes use of the triangular inequality, we transform the above to

$$\|\partial_x(u^m - v^m)\| \le \theta_1 \|u-v\| \sum_{j=0}^{m-2} C^j_{m-1}\left\|u^j v^{m-j-2}\right\|. \quad (22)$$

However to proceed with our demonstration, we must assume that u, v are bounded, implying that we can find a positive constant, say M, such that $\|u\|, \|v\| \le M$. Therefore,

$$\begin{aligned} \sum_{j=0}^{m-2} C^j_{m-1}\left\|u^j v^{m-j-2}\right\| &\le \sum_{j=0}^{m-2} C^j_{m-2}M^j M^{m-j-2} \\ &= (2M)^{m-2}. \end{aligned} \quad (23)$$

Thus, we have the following:

$$\|\partial_x(u^m - v^m)\| \le (2M)^{m-2}\theta_1\|u-v\|^2. \quad (24)$$

We shall now consider the following:

$$\left(\partial_{xxx}^3\left(u^n-v^n\right),u-v\right). \tag{25}$$

Again, using the Cauchy-Schwartz inequality, we have the following relationship:

$$\left(\partial_{xxx}^3\left(u^n-v^n\right),u-v\right)\le\left\|\partial_{xxx}^3\left(u^n-v^n\right)\right\|\|u-v\|. \tag{26}$$

Using the properties of the inner product the above can further be converted to

$$\begin{aligned}\left(\partial_{xxx}^3\left(u^n-v^n\right),u-v\right)&\le\theta_2\left\|\partial_{xxx}^3\left(u^n-v^n\right)\right\|\|u-v\|\\&\le\theta_2\theta_3\left\|\partial_{xx}^2\left(u^n-v^n\right)\right\|\|u-v\|\\&\le\theta_2\theta_3\theta_4\left\|\partial_{x}^1\left(u^n-v^n\right)\right\|\|u-v\|\\&\le\theta_2\theta_3\theta_4\theta_5\left\|\left(u^n-v^n\right)\right\|\|u-v\|.\end{aligned} \tag{27}$$

However,

$$\begin{aligned}\left\|\left(u^n-v^n\right)\right\|&=\|u-v\|\left\|\sum_{j=0}^{m-2}C_{n-2}^j u^j v^{n-j-2}\right\|\\&\le\|u-v\|\sum_{j=0}^{m-2}C_{n-2}^j\left\|u^j\right\|\|v\|^{n-j-2};\end{aligned} \tag{28}$$

using the same argument as before, we obtain

$$\begin{aligned}&\left(\partial_{xxx}^3\left(u^n-v^n\right),u-v\right)\\&\quad\le\theta_2\theta_3\theta_4\theta_5(2M)^{m-2}\|u-v\|^2.\end{aligned} \tag{29}$$

Also

$$\begin{aligned}&\mu_0\left(\partial_{xxxxxx}^6(u-v),u-v\right)\\&\quad\le\mu_0\theta_6\theta_7\theta_8\theta_9\theta_{10}\theta_{11}\|u-v\|^2,\\&\mu_1\left(\partial_{xxxxxt}^6(u-v),u-v\right)\\&\quad\le\mu_1\theta_{12}\theta_{13}\theta_{14}\theta_{15}\theta_{16}\theta_{17}\|u-v\|^2,\\&\mu_2\left(\partial_{xxxxtt}^6(u-v),u-v\right)\\&\quad\le\mu_2\theta_{18}\theta_{19}\theta_{20}\theta_{21}\theta_{22}\theta_{23}\|u-v\|^2,\\&\mu_3\left(\partial_{xxxttt}^6(u-v),u-v\right)\\&\quad\le\mu_3\theta_{24}\theta_{25}\theta_{26}\theta_{27}\theta_{28}\theta_{29}\|u-v\|^2,\\&\mu_4\left(\partial_{xxtttt}^6(u-v),u-v\right)\\&\quad\le\mu_4\theta_{30}\theta_{31}\theta_{32}\theta_{33}\theta_{34}\theta_{35}\|u-v\|^2,\\&\mu_5\left(\partial_{xttttt}^6(u-v),u-v\right)\\&\quad\le\mu_5\theta_{36}\theta_{37}\theta_{38}\theta_{39}\theta_{40}\theta_{41}\|u-v\|^2,\\&\mu_6\left(\partial_{tttttt}^6(u-v),u-v\right)\\&\quad\le\mu_6\theta_{42}\theta_{43}\theta_{44}\theta_{45}\theta_{46}\theta_{47}\|u-v\|^2.\end{aligned} \tag{30}$$

Now replacing these relations into (17), we obtain the following relation:

$$\begin{aligned}(T(u)-T(v),u-v)\ge\Big(&-(2M)^{m-2}\theta_1\\&-\theta_2\theta_3\theta_4\theta_5(2M)^{m-2}+\mu_0\theta_6\theta_7\theta_8\theta_9\theta_{10}\theta_{11}\\&-\mu_1\theta_{12}\theta_{13}\theta_{14}\theta_{15}\theta_{16}\theta_{17}+\mu_2\theta_{18}\theta_{19}\theta_{20}\theta_{21}\theta_{22}\theta_{23}\\&-\mu_3\theta_{24}\theta_{25}\theta_{26}\theta_{27}\theta_{28}\theta_{29}+\mu_4\theta_{30}\theta_{31}\theta_{32}\theta_{33}\theta_{34}\theta_{35}\\&-\mu_5\theta_{36}\theta_{37}\theta_{38}\theta_{39}\theta_{40}\theta_{41}+\mu_6\theta_{42}\theta_{43}\theta_{44}\theta_{45}\theta_{46}\theta_{47}\Big)\|u\\&-v\|^2.\end{aligned} \tag{31}$$

Take

$$\begin{aligned}\beta=\Big(&-(2M)^{m-2}\theta_1-\theta_2\theta_3\theta_4\theta_5(2M)^{m-2}\\&+\mu_0\theta_6\theta_7\theta_8\theta_9\theta_{10}\theta_{11}-\mu_1\theta_{12}\theta_{13}\theta_{14}\theta_{15}\theta_{16}\theta_{17}\\&+\mu_2\theta_{18}\theta_{19}\theta_{20}\theta_{21}\theta_{22}\theta_{23}-\mu_3\theta_{24}\theta_{25}\theta_{26}\theta_{27}\theta_{28}\theta_{29}\\&+\mu_4\theta_{30}\theta_{31}\theta_{32}\theta_{33}\theta_{34}\theta_{35}-\mu_5\theta_{36}\theta_{37}\theta_{38}\theta_{39}\theta_{40}\theta_{41}\\&+\mu_6\theta_{42}\theta_{43}\theta_{44}\theta_{45}\theta_{46}\theta_{47}\Big)\end{aligned} \tag{32}$$

such that

$$(T(u)-T(v),u-v)\ge\beta\|u-v\|^2. \tag{33}$$

The next step in this proof will be to evaluate

$$\begin{aligned}(T(u)-T(v),w)=&-\left(\partial_x\left(u^m-v^m\right),w\right)\\&-\left(\partial_{xxx}^3\left(u^n-v^n\right),w\right)\\&-\mu_0\left(\partial_{xxxxxx}^6(u-v),u-v\right)\\&-\mu_1\left(\partial_{xxxxxt}^6(u-v),w\right)\\&-\mu_2\left(\partial_{xxxxtt}^6(u-v),w\right)\\&-\mu_3\left(\partial_{xxxttt}^6(u-v),w\right)\\&-\mu_4\left(\partial_{xxtttt}^6(u-v),w\right)\\&-\mu_5\left(\partial_{xttttt}^6(u-v),w\right)\\&-\mu_6\left(\partial_{tttttt}^6(u-v),w\right).\end{aligned} \tag{34}$$

Nevertheless, following the discussion presented, we shall have

$$(T(u)-T(v),w)\ge k\|u-v\|\|w\|, \tag{35}$$

where k is defined as

$$\begin{aligned}k=\Big(&-(2M)^{m-2}\theta_1-\theta_2\theta_3\theta_4\theta_5(2M)^{m-2}\\&+\mu_0\theta_6\theta_7\theta_8\theta_9\theta_{10}\theta_{11}-\mu_1\theta_{12}\theta_{13}\theta_{14}\theta_{15}\theta_{16}\theta_{17}\\&+\mu_2\theta_{18}\theta_{19}\theta_{20}\theta_{21}\theta_{22}\theta_{23}-\mu_3\theta_{24}\theta_{25}\theta_{26}\theta_{27}\theta_{28}\theta_{29}\\&-\mu_4\theta_{30}\theta_{31}\theta_{32}\theta_{33}\theta_{34}\theta_{35}-\mu_5\theta_{36}\theta_{37}\theta_{38}\theta_{39}\theta_{40}\theta_{41}\\&-\mu_6\theta_{42}\theta_{43}\theta_{44}\theta_{45}\theta_{46}\theta_{47}\Big).\end{aligned} \tag{36}$$

This completes the proof. □

Our next concern will consist of investigating on the uniqueness of the special solution. We shall assume that there exist two special solutions, say u_1 and u_2, satisfying (2); then, using (33), we have

$$\left(T\left(u_1\right)-T\left(u_2\right),u_1-u_2\right)\geq\beta\left\|u_1-u_2\right\|^2. \tag{37}$$

Since both solutions satisfy (2), thus $T(u_1)-T(u_2)\cong 0$; then, $(T(u_1)-T(u_2),u_1-u_2)\cong 0$. Nonetheless,

$$\left\|u_1-u_2\right\|^2=\left\|u_1-u_2\right\|\left\|u_1-u_2\right\|. \tag{38}$$

Thus if u is the exact solution of (2) we have the following relation,

$$\left\|u_1-u+u-u_2\right\|\leq\left\|u-u_2\right\|+\left\|u-u_2\right\|<\frac{\varepsilon}{2\beta}+\frac{\varepsilon}{2\beta} \tag{39}$$

with ε being a very small positive parameter closer to zero. Then the inequality (37) can be converted to

$$0\geq\varepsilon\left\|u_1-u_2\right\|. \tag{40}$$

And this implied, with the benefits of the norm, that

$$\left\|u_1-u_2\right\|=0\Longrightarrow u_1-u_2. \tag{41}$$

3.1. Application of the Scheme. We shall present the special solutions for some examples using the scheme presented earlier.

Example 1. Here we assume that the perturbation fractional $K(m,m)$ Rosenau-Hyman equation is such that $|\mu_j|=0$; then we have the following equation:

$$\partial_t^{\alpha}u+\partial_x u^m+\partial_{xxx}^3 u^m=0. \tag{42}$$

Using the presented scheme and also using the fractional integral proposed as antiderivative of the conformable fractional derivative, we obtain the following iteration formula:

$$\begin{aligned}u_{k+1}&=\mathscr{L}^{-1}\left(-\frac{1}{s^{\alpha}}\right.\\&\left.\cdot\mathscr{L}\left(\frac{1}{\Gamma(\alpha)}\int_0^t(l)^{\alpha-1}\left(-\partial_x {u^m}_k-\partial_{xxx}^3 {u^m}_k\right)dl\right)\right),\\u_0(x,t)&=\left(\frac{2m}{m+1}\sin^2\left(\frac{m-1}{2m}(x)\right)\right)^{1/n-1}.\end{aligned} \tag{43}$$

Now using the algorithm, we reach the following:

$$\begin{aligned}&u_1(x,t)\\&\quad=-\frac{t^{2\alpha}\left(-4+(2-3m)m+3m^2\cos[(-1+m)x/m]\right)\cot[(-1+m)x/2m]\csc[(-1+m)x/2m]^2\left(2^{1/(-1+m)}(m\sin[(-1+m)x/2m]/(1+m))^{1/(-1+m)}\right)^m}{16m^2\Gamma[1+2\alpha]},\\&u_2(x,t)=\left(2^{-4(2+m)}(-2+m)t^{2\alpha+2m\alpha}\left(24576-18432m+33024m^2-199296m^3+432768m^4-540576m^5+500784m^6-358992m^7+195624m^8\right.\right.\\&\quad-73332m^9+12474m^{10}-8\left(-2048-8192m+32640m^2-47072m^3+56392m^4-73464m^5+81546m^6-65784m^7+38151m^8-14877m^9+2673m^{10}\right)\\&\quad\cdot\cos\left[\frac{(-1+m)x}{m}\right]+\left(8192-6144m-21504m^2-51712m^3+198912m^4-276864m^5+263616m^6-217152m^7+141984m^8-62262m^9+13365m^{10}\right)\\&\quad\cdot\cos\left[\frac{2(-1+m)x}{m}\right]-9216m^2\cos\left[\frac{3(-1+m)x}{m}\right]+256m^3\cos\left[\frac{3(-1+m)x}{m}\right]+36128m^4\cos\left[\frac{3(-1+m)x}{m}\right]-30816m^5\cos\left[\frac{3(-1+m)x}{m}\right]\\&\quad-10952m^6\cos\left[\frac{3(-1+m)x}{m}\right]+38816m^7\cos\left[\frac{3(-1+m)x}{m}\right]-36108m^8\cos\left[\frac{3(-1+m)x}{m}\right]+19332m^9\cos\left[\frac{3(-1+m)x}{m}\right]-5940m^{10}\\&\quad\cdot\cos\left[\frac{3(-1+m)x}{m}\right]+768m^2\cos\left[\frac{4(-1+m)x}{m}\right]-4992m^3\cos\left[\frac{4(-1+m)x}{m}\right]+16512m^4\cos\left[\frac{4(-1+m)x}{m}\right]-21216m^5\cos\left[\frac{4(-1+m)x}{m}\right]\\&\quad+12816m^6\cos\left[\frac{4(-1+m)x}{m}\right]-4464m^7\cos\left[\frac{4(-1+m)x}{m}\right]+3384m^8\cos\left[\frac{4(-1+m)x}{m}\right]-2700m^9\cos\left[\frac{4(-1+m)x}{m}\right]+1782m^{10}\\&\quad\cdot\cos\left[\frac{4(-1+m)x}{m}\right]+288m^4\cos\left[\frac{5(-1+m)x}{m}\right]-864m^5\cos\left[\frac{5(-1+m)x}{m}\right]+792m^6\cos\left[\frac{5(-1+m)x}{m}\right]-864m^7\cos\left[\frac{5(-1+m)x}{m}\right]+324m^8\\&\quad\left.\cdot\cos\left[\frac{5(-1+m)x}{m}\right]-108m^9\cos\left[\frac{5(-1+m)x}{m}\right]-324m^{10}\cos\left[\frac{5(-1+m)x}{m}\right]+54m^9\cos\left[\frac{6(-1+m)x}{m}\right]+27m^{10}\cos\left[\frac{6(-1+m)x}{m}\right]\right)\\&\quad\left.\cdot\csc\left[\frac{(-1+m)x}{m}\right]^3\text{Gamma}[1+\alpha+2m\alpha]\left(-\frac{\left(-4+2m-3m^2+3m^2\cos[(-1+m)x/m]\right)\cot[(-1+m)x/2m]\csc[(-1+m)x/2m]^2\left(2^{1/(-1+m)}(m\sin[(-1+m)x/2m]/(1+m))^{1/(-1+m)}\right)^m}{m^2\text{Gamma}[1+2\alpha]}\right)^m\right).\end{aligned} \tag{44}$$

Using the iteration formula the remaining terms can be calculated; however, if we go to infinity, the special solution for this version can be given as

$$u_{spe}(x,t)=\left(\frac{2m}{m+1}\left(\sin\left[\frac{m-1}{2m}x\right]\sum_{k=0}^{\infty}\frac{(-1)^k\left(((m-1)/2m)t\right)^{2\alpha}}{\Gamma(1+2\alpha k)}+\cos\left[\frac{m-1}{2m}x\right]\sum_{k=0}^{\infty}\frac{(-1)^k\left(((m-1)/2m)t\right)^{2\alpha+1}}{\Gamma(2\alpha k+1)}\right)^2\right)^{1/m-1}, \tag{45}$$

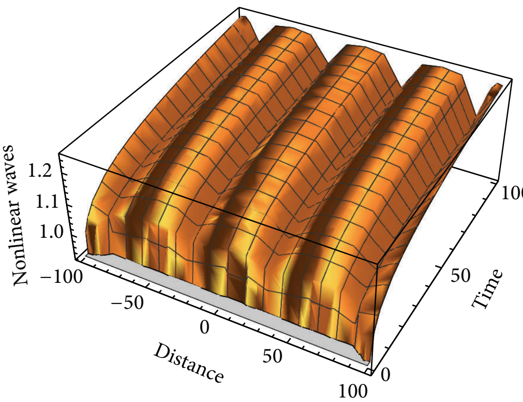

FIGURE 1: The special solution for alpha = 0.5 and $m = 50$.

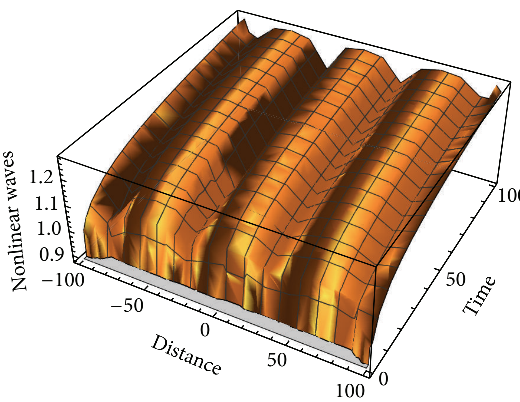

FIGURE 2: The special solution for alpha = 0.55 and $m = 50$.

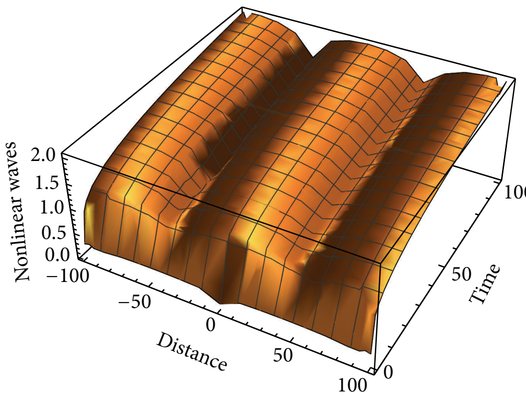

FIGURE 3: The special solution for alpha = 0.90 and $m = 20$.

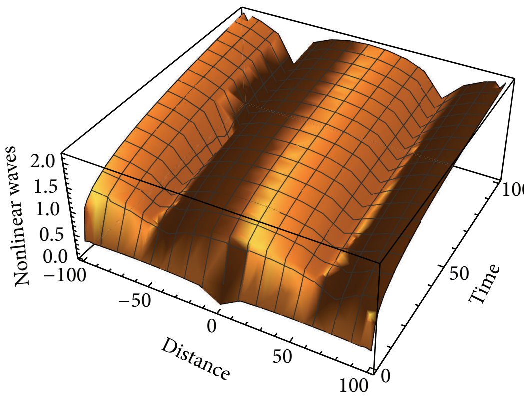

FIGURE 4: The special solution for alpha = 0.95 and $m = 20$.

which of course satisfies that, for alpha equal to 1, we obtain the exact solution of the $K(m, m)$ Rosenau-Hyman equation

$$u(x,t) = \left(\frac{2m}{m+1}\sin^2\left(\frac{m-1}{2m}(x+t)\right)\right)^{1/n-1}. \quad (46)$$

We shall present the numerical results in Figures 1–4 for different value of alpha and m. We have depicted the numerical results for different alpha and m in Figures 1, 2, 3, and 4; for example, if $m = 50$ and alpha = 0.5, we have Figure 1.

For example if alpha is 0.55 and m is 50 we will observe nonlinear wave as represented in Figure 2.

Figures 1–4 as a result of introduction of fractional order derivative help us to understand better the role of nonlinear dispersion in prototype configuration. The solitary wave solutions of these equations have extraordinary assets according to the variation of the order of fractional derivative together with m; in addition the solitary wave collides elastically and they have compact support. The figures also reveal that, when two compactons collide, the interface location is indicated by the beginning of low-amplitude compacton-anticompacton pairs. These equations seem to have only a finite number of local conservation laws [1]. Nonetheless, the behaviour and the stability of these compactons are very similar to those observed in completely integrable systems [1].

4. Conclusion

One of the most difficult tasks in the area of applied mathematics is perhaps to make use of mathematical equations in order to explain adequately the physical phenomenon. Partial differential equations have been intensively used for this purpose in the past decades. However researchers have encountered some limitation by modelling physical phenomenon using the integer derivative order. In the way of trying to extend these limitations, the concept of noninteger order derivative has been introduced. In the contemporary day numerous objective occurrences were clarified with pronounced accomplishment in the light of the perception of noninteger order derivatives. Exclusively, the compensations of fractional calculus and fractional order replicas and their diligences in the field of nonlinear wave motion have beforehand been intensively reexamined throughout the last few epochs with exceptional conclusion. We have therefore investigated within that scope of fractional calculus the nonlinear wave phenomenon that is usually described via the Perturbation $K(n, m)$ Rosenau-Hyman equation. We derived the special solution of the extended equation using the so-called Laplace transform and the Lagrange multiplier. We proved with great success the convergence of this method for solving the extended equation. On the other hand we showed the uniqueness of the special solution using some properties of the inner product within a well-constructed Hilbert space. Some numerical results were depicted from Figures 1–4. The special solution seems to describe the real world problem better than the standard solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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